SET-THEORETIC METHODS IN TOPOLOGY AND REAL FUNCTIONS THEORY

DEDICATED TO 80TH BIRTHDAY OF LEV BUKOVSKÝ

9.9. - 13.9.2019, Košice, Slovakia

CONFIRMED SPEAKERS

Aleksander Błaszczyk
Vera Fischer
István Juhász
Menachem Magidor
Dilip Raghavan

SCIENTIFIC COMMITTEE

David Chodounský
Martin Goldstern
Klaas Pieter Hart
Ľubica Holá
Thomas Jech
Miroslav Repický
Masami Sakai
Lyubomyr Zdomskyy

ORGANIZING COMMITTEE

Peter Eliaš Miroslav Repický Viera Šottová Jaroslav Šupina

umv.science.upjs.sk/setmath
setmath@upjs.sk

supported by
Visegrad Fund

On a level measure

Jaroslav Šupina

joint work with J. Borzová, L. Halčinová, O. Hutník and J. Kiseľák

Institute of Mathematics
Faculty of Science
P.J. Šafárik University in Košice

February 1, 2019

Supported by the grant APVV-16-0337.

Basic setting

X is a topological space.

$\mathbf{E}_{\mathrm{B}}$ is a family of all Borel subsets of X.

Warning!

measure $\neq$ measure

measure = function $m : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$ such that $m(\emptyset) = 0$

monotone measure

non-additive integrals

Warning!

measure $\neq$ measure

measure = function $m : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$ such that $m(\emptyset) = 0$

monotone measure

non-additive integrals

Warning!

measure $\neq$ measure

measure = function $m : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$ such that $m(\emptyset) = 0$

monotone measure

non-additive integrals

Warning!

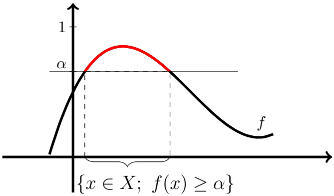
measure $\neq$ measure

measure = function $m : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$ such that $m(\emptyset) = 0$

monotone measure

non-additive integrals

Level sets and level measure

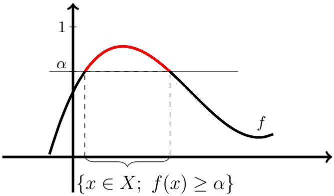


$$h_{\mu,f}(\alpha) = \mu(f \geq \alpha) := \mu(\{x \in X;\ f(x) \geq \alpha\})$$

Integrals

- The Lebesgue integral $\quad (\mathrm{L}) \int_X f \, \mathrm{d}\mu = \int_0^\infty \mu(f \geq \alpha) \, \mathrm{d}\alpha$
- The Choquet integral $\quad (\mathrm{Ch}) \int_X f \, \mathrm{d}\mu = \int_0^\infty \mu(f \geq \alpha) \, \mathrm{d}\alpha$
- The Sugeno integral $\quad (\mathrm{Su}) \int_X f \, \mathrm{d}\mu = \sup_{\alpha > 0} M \{\alpha, \mu(f \geq \alpha)\}$

Level sets and level measure

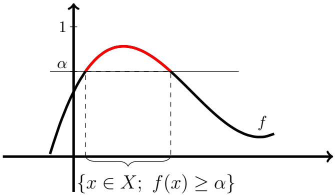


$$h_{\mu,f}(\alpha) = \mu(f \geq \alpha) := \mu(\{x \in X;\ f(x) \geq \alpha\})$$

Integrals

- The Lebesgue integral $\quad (\mathrm{L}) \int_X f \, \mathrm{d}\mu = \int_0^\infty \mu(f \geq \alpha) \, \mathrm{d}\alpha$
- The Choquet integral $\quad (\mathrm{Ch}) \int_X f \, \mathrm{d}\mu = \int_0^\infty \mu(f \geq \alpha) \, \mathrm{d}\alpha$
- The Sugeno integral $\quad (\mathrm{Su}) \int_X f \, \mathrm{d}\mu = \sup_{\alpha > 0} M \{\alpha, \mu(f \geq \alpha)\}$

Level sets and level measure

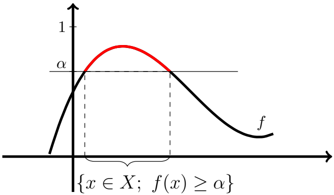


$$h_{\mu,f}(\alpha) = \mu(f \geq \alpha) := \mu(\{x \in X;\ f(x) \geq \alpha\})$$

Integrals

- The Lebesgue integral $\quad (\mathrm{L}) \int\limits_X f \, \mathrm{d}\mu = \int\limits_0^\infty \mu(f \geq \alpha) \, \mathrm{d}\alpha$
- The Choquet integral $\quad (\mathrm{Ch}) \int\limits_X f \, \mathrm{d}\mu = \int\limits_0^\infty \mu(f \geq \alpha) \, \mathrm{d}\alpha$
- The Sugeno integral $\quad (\mathrm{Su}) \int\limits_X f \, \mathrm{d}\mu = \sup\limits_{\alpha > 0} M\,\{\alpha, \mu(f \geq \alpha)\}$

Level sets and level measure



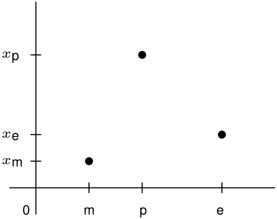
$$h_{\mu,f}(\alpha) = \mu(f \geq \alpha) := \mu(\{x \in X;\ f(x) \geq \alpha\})$$

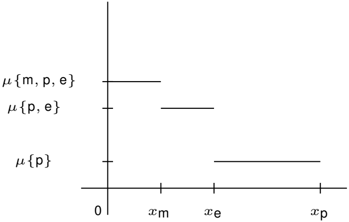
Integrals

- The Lebesgue integral $(\mathrm{L}) \int\limits_X f \, \mathrm{d}\mu = \int\limits_0^\infty \mu(f \geq \alpha) \, \mathrm{d}\alpha$
- The Choquet integral $(\mathrm{Ch}) \int\limits_X f \, \mathrm{d}\mu = \int\limits_0^\infty \mu(f \geq \alpha) \, \mathrm{d}\alpha$
- The Sugeno integral $(\mathrm{Su}) \int\limits_X f \, \mathrm{d}\mu = \sup\limits_{\alpha > 0} M \{\alpha, \mu(f \geq \alpha)\}$

Choquet, G.: *Theory of capacities*, Ann. Inst. Fourier 5 (1953), 131–295.

$$\text{(Ch)} \int_X h \,\mathrm{d}\mu := \int_0^\infty \mu\left(\{x \in X : h(x) > \alpha\}\right)\,\mathrm{d}\alpha$$



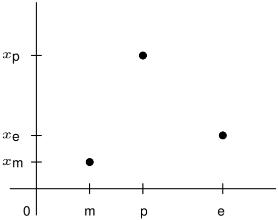


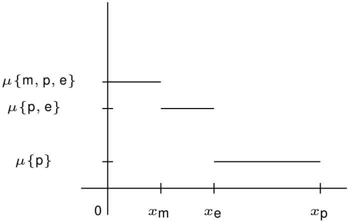
$$\text{(Ch)} \int_{\{\text{m,p,e}\}} x_i \,\mathrm{d}\mu = x_\text{m} \cdot \mu\{\text{m,p,e}\} + (x_\text{e} - x_\text{m}) \cdot \mu\{\text{p,e}\} + (x_\text{p} - x_\text{e}) \cdot \mu\{\text{p}\}$$

$$= x_\text{m} \cdot \Big(\mu\{\text{m,p,e}\} - \mu\{\text{p,e}\}\Big) + x_\text{e} \cdot \Big(\mu\{\text{p,e}\} - \mu\{\text{p}\}\Big) + x_\text{p} \cdot \mu\{\text{p}\}$$

Choquet, G.: *Theory of capacities*, Ann. Inst. Fourier 5 (1953), 131–295.

$$\text{(Ch)} \int_X h \, \mathrm{d}\mu := \int_0^\infty \mu\left(\{x \in X : h(x) > \alpha\}\right) \, \mathrm{d}\alpha$$





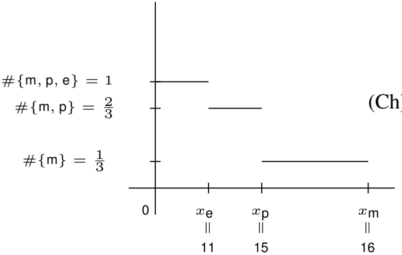
$$\text{(Ch)} \int_{\{\mathsf{m},\mathsf{p},\mathsf{e}\}} x_i \, \mathrm{d}\mu = x_\mathsf{m} \cdot \mu\{\mathsf{m}, \mathsf{p}, \mathsf{e}\} + (x_\mathsf{e} - x_\mathsf{m}) \cdot \mu\{\mathsf{p}, \mathsf{e}\} + (x_\mathsf{p} - x_\mathsf{e}) \cdot \mu\{\mathsf{p}\}$$

$$= x_\mathsf{m} \cdot \Big(\mu\{\mathsf{m}, \mathsf{p}, \mathsf{e}\} - \mu\{\mathsf{p}, \mathsf{e}\}\Big) + x_\mathsf{e} \cdot \Big(\mu\{\mathsf{p}, \mathsf{e}\} - \mu\{\mathsf{p}\}\Big) + x_\mathsf{p} \cdot \mu\{\mathsf{p}\}$$

GRABISCH, M.: The application of fuzzy integrals in multicriteria decision making. *European J. Oper. Res.* **89** (1996), 445–456.

	mathematics	physics	english	mean
1	16	15	11	14

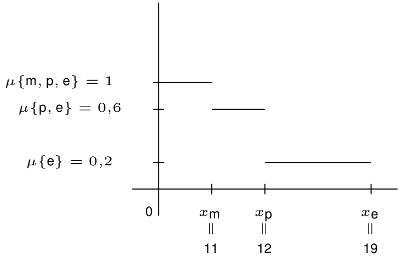
	{m}	{p}	{e}	{m, p}	{m, e}	{p, e}	{m, p, e}
#	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	1



$$\text{(Ch)} \int_{\{m,p,e\}} x_i \, d\# = \frac{1}{3} \cdot 11 + \frac{1}{3} \cdot 15 + \frac{1}{3} \cdot 16$$

	mathematics	physics	english	mean	μ-mean
1	16	15	11	14	14,6
2	11	12	19	14	13

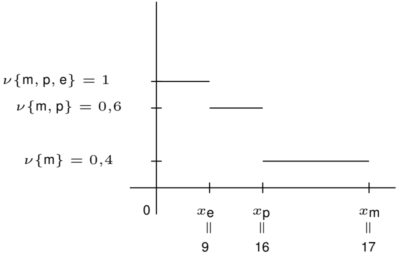
	{m}	{p}	{e}	{m, p}	{m, e}	{p, e}	{m, p, e}
μ	0,4	0,4	0,2	0,8	0,6	0,6	1



(Ch) $\displaystyle\int_{\{\text{m,p,e}\}} x_i \, \mathrm{d}\mu = 0{,}2 \cdot 19 + 0{,}4 \cdot 12 + 0{,}4 \cdot 11$

	mathematics	physics	english	mean	μ-mean	ν-Choquet
1	16	15	11	14	14,6	13,8
2	11	12	19	14	13	13,3
3	17	16	9	14	15,5	13,6

	$\{m\}$	$\{p\}$	$\{e\}$	$\{m,p\}$	$\{m,e\}$	$\{p,e\}$	$\{m,p,e\}$
ν	0,4	0,4	0,2	0,6	0,9	0,9	1



A new concept - three steps to super level measures

metric space X	premeasure $\sigma : \mathbf{E} \subseteq \mathbf{E}_{\mathrm{B}} \to [0, \infty)$

(A) **Size.** a function $\mathsf{s} : \mathcal{B}(X) \to [0, +\infty]^{\mathbf{E}}$ satisfying

(i) if $|f| \leq |g|$, then $\mathsf{s}(f)(a) \leq \mathsf{s}(g)(a)$;

(ii) $\mathsf{s}(\lambda f)(a) = |\lambda| \, \mathsf{s}(f)(a)$ for each $\lambda \in \mathbb{C}$;

(iii) $\mathsf{s}(f + g)(a) \leq C_{\mathsf{s}} \, \mathsf{s}(f)(a) + C_{\mathsf{s}} \, \mathsf{s}(g)(a)$ for some fixed $C_{\mathsf{s}} \geq 1$ depending only on s.

A triple (X, σ, s) is called an outer measure space.

(B) **Outer essential supremum.** $f \in \mathcal{B}(X)$, $b \in \mathbf{E}_{\mathrm{B}}$

$$\operatorname*{outsup}_{b} \mathsf{s}(f) := \sup\{\mathsf{s}(f \mathbf{1}_b)(a);\ a \in \mathbf{E}\}$$

(C) **Super level measure.** (X, σ, s), $f \in \mathcal{B}(X)$, $\alpha > 0$

$$\mu(\mathsf{s}(f) > \alpha) := \inf \left\{ \mu(b) : \ b \in \mathbf{E}_{\mathrm{B}}, \ \operatorname*{outsup}_{X \setminus b} \mathsf{s}(f) \leq \alpha \right\}$$

Is a new concept useful?

DO Y., THIELE C., L^p theory for outer measures and two themes of Lennart Carleson united, *Bull. Amer. Math. Sci.* **52** (2) (2015), 249–296.

- natural L^p theory for outer measures offers a unifying language for both Carleson measure and time-freqency analysis
- gaining a streamlined view on time-frequency analysis was the original motivation for their paper
- the outcome of a long evolution process
- a point of the paper is that in many examples of their interest the bound is a Hölder inequality with respect to an outer measure

HALČINOVÁ L., HUTNÍK O., KISEĽÁK J., ŠUPINA J., *Beyond the scope of super level measures*, Fuzzy Sets and Systems, https://doi.org/10.1016/j.fss.2018.03.007.

Is a new concept useful outside of functional analysis?

Hopefully . . .

Three steps to super level measures - Y. Do and C. Thiele, modified

topological space X | ~~metric space X~~ | ~~premeasure $\sigma : \mathbf{E} \subseteq \mathbf{E}_{\rm B} \to [0,\infty)$~~

monotone measure $\mu : \mathbf{E}_{\rm B} \to [0,\infty)$

(A) **Size.** a function $\mathsf{s} : \mathcal{B}(X) \to [0,+\infty]^{\mathbf{E}}$ satisfying ($\mathbf{E} \to \mathbf{E}_{\rm B}$)

(i) if $|f| \le |g|$, then $\mathsf{s}(f)(a) \le \mathsf{s}(g)(a)$;

(ii) $\mathsf{s}(\lambda f)(a) = |\lambda|\, \mathsf{s}(f)(a)$ for each $\lambda \in \mathbb{C}$;

(iii) $\mathsf{s}(f+g)(a) \le C_{\mathsf{s}}\, \mathsf{s}(f)(a) + C_{\mathsf{s}}\, \mathsf{s}(g)(a)$ for some fixed $C_{\mathsf{s}} \ge 1$ depending only on s.

A triple (X, σ, s) is called an outer measure space.

$(X, \mathbf{E}, \mathsf{s})$ a sub-Borel size space

(B) **Outer essential supremum.** $f \in \mathcal{B}(X)$, $F \in \mathbf{E}_{\rm B}$

$$\operatorname*{outsup}_{b} \mathsf{s}(f)\langle \mathbf{E} \rangle \qquad \operatorname*{outsup}_{b} \mathsf{s}(f) := \sup\{\mathsf{s}(f \mathbf{1}_b)(a) : \ a \in \mathbf{E}\}$$

(C) **Super level measure.** (X, σ, s), $f \in \mathcal{B}(X)$, $\alpha > 0$ $\quad (X, \mathbf{E}, \mathsf{s})$

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) \qquad \mu(\mathsf{s}(f) > \alpha) := \inf \left\{ \mu(b) : \ b \in \mathbf{E}_{\rm B}, \ \operatorname*{outsup}_{X \setminus b} \mathsf{s}(f) \le \alpha \right\}$$

An example

$$\boxed{\mathsf{s}_{\mathsf{int}}(f)(a) = (\mathrm{L}) \int_a |f| \, \mathrm{d}\mu}$$

(A) **Size.** a function $\mathsf{s} : \mathcal{B}(X) \to [0, +\infty]^{\mathbf{E}_{\mathrm{B}}}$ satisfying

(i) if $|f| \leq |g|$, then $\mathsf{s}(f)(a) \leq \mathsf{s}(g)(a)$;

(ii) $\mathsf{s}(\lambda f)(a) = |\lambda| \, \mathsf{s}(f)(a)$ for each $\lambda \in \mathbb{C}$;

(iii) $\mathsf{s}(f + g)(a) \leq C_{\mathsf{s}} \, \mathsf{s}(f)(a) + C_{\mathsf{s}} \, \mathsf{s}(g)(a)$ for some fixed $C_{\mathsf{s}} \geq 1$ depending only on s.

An example

$$\boxed{\mathsf{s}_{\mathsf{int}}(f)(a) = (\mathrm{L}) \int_a |f| \,\mathrm{d}\mu}$$

(B) **Outer essential supremum.** $f \in \mathcal{B}(X)$, $b \in \mathbf{E}_{\mathrm{B}}$, reasonable $\mathbf{E}$

$$\begin{aligned}\operatorname*{outsup}_b \mathsf{s}_{\mathsf{int}}(f)\langle\mathbf{E}\rangle &:= \sup\{\mathsf{s}_{\mathsf{int}}(f\mathbf{1}_b)(a) :\ a \in \mathbf{E}\} \\ &= \sup\{(\mathrm{L}) \int_{a\cap b} f \,\mathrm{d}\mu :\ a \in \mathbf{E}\} \\ &= (\mathrm{L}) \int_b f \,\mathrm{d}\mu = \mathsf{s}_{\mathsf{int}}(f)(b)\end{aligned}$$

An example

$$\boxed{\mathsf{s}_{\mathsf{int}}(f)(a) = (\mathrm{L}) \int_a |f| \,\mathrm{d}\mu}$$

(B) **Outer essential supremum.** $f \in \mathcal{B}(X)$, $b \in \mathbf{E}_{\mathrm{B}}$, reasonable $\mathbf{E}$

$$\operatorname*{outsup}_b \mathsf{s}_{\mathsf{int}}(f)\langle \mathbf{E} \rangle := \sup\{\mathsf{s}_{\mathsf{int}}(f\mathbf{1}_b)(a) : \ a \in \mathbf{E}\}$$

$$= \sup\{(\mathrm{L}) \int_{a \cap b} f \,\mathrm{d}\mu : \ a \in \mathbf{E}\}$$

$$= (\mathrm{L}) \int_b f \,\mathrm{d}\mu = \mathsf{s}_{\mathsf{int}}(f)(b)$$

An example

$$\boxed{\mathsf{s}_{\mathsf{int}}(f)(a) = (\mathrm{L}) \int_a |f| \,\mathrm{d}\mu}$$

(B) **Outer essential supremum.** $f \in \mathcal{B}(X)$, $b \in \mathbf{E}_{\mathrm{B}}$, reasonable $\mathbf{E}$

$$\operatorname*{outsup}_{b} \mathsf{s}_{\mathsf{int}}(f)\langle \mathbf{E} \rangle := \sup\{\mathsf{s}_{\mathsf{int}}(f\mathbf{1}_b)(a) : \; a \in \mathbf{E}\}$$

$$= \sup\{(\mathrm{L}) \int_{a \cap b} f \,\mathrm{d}\mu : \; a \in \mathbf{E}\}$$

$$= (\mathrm{L}) \int_b f \,\mathrm{d}\mu = \mathsf{s}_{\mathsf{int}}(f)(b)$$

An example

$$\boxed{\mathsf{s}_{\mathsf{int}}(f)(a) = (\mathrm{L}) \int_a |f| \,\mathrm{d}\mu}$$

(B) **Outer essential supremum.** $f \in \mathcal{B}(X)$, $b \in \mathbf{E}_{\mathrm{B}}$, reasonable $\mathbf{E}$

$$\begin{aligned}\operatorname*{outsup}_b \mathsf{s}_{\mathsf{int}}(f)\langle\mathbf{E}\rangle &:= \sup\{\mathsf{s}_{\mathsf{int}}(f\mathbf{1}_b)(a) : \ a \in \mathbf{E}\} \\ &= \sup\{(\mathrm{L}) \int_{a\cap b} f \,\mathrm{d}\mu : \ a \in \mathbf{E}\} \\ &= (\mathrm{L}) \int_b f \,\mathrm{d}\mu = \mathsf{s}_{\mathsf{int}}(f)(b)\end{aligned}$$

An example

$$\boxed{\mathsf{s}_{\mathsf{int}}(f)(a) = (\mathrm{L}) \int_a |f| \,\mathrm{d}\mu}$$

(B) **Outer essential supremum.** $f \in \mathcal{B}(X)$, $b \in \mathbf{E}_{\mathrm{B}}$, reasonable $\mathbf{E}$

$$\begin{aligned} \operatorname*{outsup}_b \mathsf{s}_{\mathsf{int}}(f)\langle \mathbf{E} \rangle &:= \sup\{\mathsf{s}_{\mathsf{int}}(f\mathbf{1}_b)(a) : \; a \in \mathbf{E}\} \\ &= \sup\{(\mathrm{L}) \int_{a \cap b} f \,\mathrm{d}\mu : \; a \in \mathbf{E}\} \\ &= (\mathrm{L}) \int_b f \,\mathrm{d}\mu = \mathsf{s}_{\mathsf{int}}(f)(b) \end{aligned}$$

An example

$$\boxed{\mathsf{s}_{\mathsf{int}}(f)(a) = (\mathrm{L}) \int_a |f| \, \mathrm{d}\mu}$$

(C) **Super level measure.** $(X, \mathbf{E}, \mathsf{s})$, $f \in \mathcal{B}(X)$, $\alpha > 0$

$$\mu(\mathsf{s}_{\mathsf{int}}(f)\langle \mathbf{E} \rangle > \alpha) := \inf \left\{ \mu(b) : \ b \in \mathbf{E}_{\mathrm{B}}, \ \operatorname*{outsup}_{X \setminus b} \mathsf{s}_{\mathsf{int}}(f) \le \alpha \right\}$$

$$= \inf \left\{ \mu(b) : \ b \in \mathbf{E}_{\mathrm{B}}, \ (\mathrm{L}) \int_{X \setminus b} f \, \mathrm{d}\mu \le \alpha \right\}$$

An example

$$\boxed{\mathsf{s}_{\mathsf{int}}(f)(a) = (\mathrm{L}) \int_a |f| \,\mathrm{d}\mu}$$

(C) **Super level measure.** $(X, \mathbf{E}, \mathsf{s})$, $f \in \mathcal{B}(X)$, $\alpha > 0$

$$\mu(\mathsf{s}_{\mathsf{int}}(f)\langle \mathbf{E}\rangle > \alpha) := \inf \left\{ \mu(b) : \ b \in \mathbf{E}_{\mathrm{B}}, \ \operatorname*{outsup}_{X \setminus b} \mathsf{s}_{\mathsf{int}}(f) \leq \alpha \right\}$$

$$= \inf \left\{ \mu(b) : \ b \in \mathbf{E}_{\mathrm{B}}, \ (\mathrm{L}) \int_{X \setminus b} f \,\mathrm{d}\mu \leq \alpha \right\}$$

An example

$$\boxed{\mathsf{s}_{\mathsf{int}}(f)(a) = (\mathrm{L}) \int\limits_a |f| \, \mathrm{d}\mu}$$

(C) **Super level measure.** $(X, \mathbf{E}, \mathsf{s})$, $f \in \mathcal{B}(X)$, $\alpha > 0$

$$\mu(\mathsf{s}_{\mathsf{int}}(f)\langle \mathbf{E} \rangle > \alpha) := \inf \left\{ \mu(b) : \ b \in \mathbf{E}_{\mathrm{B}}, \ \operatorname*{outsup}_{X \setminus b} \mathsf{s}_{\mathsf{int}}(f) \leq \alpha \right\}$$

$$= \inf \left\{ \mu(b) : \ b \in \mathbf{E}_{\mathrm{B}}, \ (\mathrm{L}) \int\limits_{X \setminus b} f \, \mathrm{d}\mu \leq \alpha \right\}$$

Is the new concept a generalization of the original one?

$$\boxed{\mathsf{s}_\infty(f)(a) = \sup |f|[a]}$$

(A) **Size.** a function $\mathsf{s} : \mathcal{B}(X) \to [0, +\infty]^{\mathbf{E}_{\mathrm{B}}}$ satisfying

(i) if $|f| \leq |g|$, then $\mathsf{s}(f)(a) \leq \mathsf{s}(g)(a)$;

(ii) $\mathsf{s}(\lambda f)(a) = |\lambda| \, \mathsf{s}(f)(a)$ for each $\lambda \in \mathbb{C}$;

(iii) $\mathsf{s}(f + g)(a) \leq C_{\mathsf{s}} \, \mathsf{s}(f)(a) + C_{\mathsf{s}} \, \mathsf{s}(g)(a)$ for some fixed $C_{\mathsf{s}} \geq 1$ depending only on s.

Is the new concept a generalization of the original one?

$$\boxed{\mathsf{s}_\infty(f)(a) = \sup |f|[a]}$$

(B) **Outer essential supremum.** $f \in \mathcal{B}(X)$, $b \in \mathbf{E}_\mathrm{B}$, reasonable $\mathbf{E}$

$$\operatorname*{outsup}_b \mathsf{s}_\infty(f)\langle \mathbf{E}\rangle := \sup\{\mathsf{s}_\infty(f\mathbf{1}_b)(a) :\ a \in \mathbf{E}\}$$

$$= \sup\{\sup |f|[a \cap b] :\ a \in \mathbf{E}\}$$

$$= \sup |f|[b] = \mathsf{s}_\infty(f)(b)$$

Is the new concept a generalization of the original one?

$$\boxed{\mathsf{s}_\infty(f)(a) = \sup |f|[a]}$$

(B) **Outer essential supremum.** $f \in \mathcal{B}(X)$, $b \in \mathbf{E}_{\mathrm{B}}$, reasonable $\mathbf{E}$

$$\operatorname*{outsup}_b \mathsf{s}_\infty(f)\langle \mathbf{E} \rangle := \sup\{\mathsf{s}_\infty(f\mathbf{1}_b)(a) : \ a \in \mathbf{E}\}$$

$$= \sup\{\sup |f|[a \cap b] : \ a \in \mathbf{E}\}$$

$$= \sup |f|[b] = \mathsf{s}_\infty(f)(b)$$

Is the new concept a generalization of the original one?

$$\boxed{\mathsf{s}_\infty(f)(a) = \sup |f|[a]}$$

(B) **Outer essential supremum.** $f \in \mathcal{B}(X)$, $b \in \mathbf{E}_{\mathrm{B}}$, reasonable $\mathbf{E}$

$$\underset{b}{\mathrm{outsup}}\, \mathsf{s}_\infty(f)\langle \mathbf{E} \rangle := \sup\{\mathsf{s}_\infty(f\mathbf{1}_b)(a) :\ a \in \mathbf{E}\}$$

$$= \sup\{\sup |f|[a \cap b] :\ a \in \mathbf{E}\}$$

$$= \sup |f|[b] = \mathsf{s}_\infty(f)(b)$$

Is the new concept a generalization of the original one?

$$\boxed{\mathsf{s}_\infty(f)(a) = \sup |f|[a]}$$

(C) **Super level measure.** $(X, \mathbf{E}, \mathsf{s})$, $f \in \mathcal{B}(X)$, $\alpha > 0$

$$\mu(\mathsf{s}_{\mathsf{int}}(f)\langle \mathbf{E}\rangle > \alpha) := \inf \left\{ \mu(b) : \ b \in \mathbf{E}_{\mathrm{B}}, \ \operatorname*{outsup}_{X \setminus b} \mathsf{s}_\infty(f) \leq \alpha \right\}$$

$$= \inf \{ \mu(b) : \ b \in \mathbf{E}_{\mathrm{B}}, \ \sup |f|[b] \leq \alpha \}$$

Is the new concept a generalization of the original one?

$$\boxed{\mathsf{s}_\infty(f)(a) = \sup |f|[a]}$$

(C) **Super level measure.** $(X, \mathbf{E}, \mathsf{s})$, $f \in \mathcal{B}(X)$, $\alpha > 0$

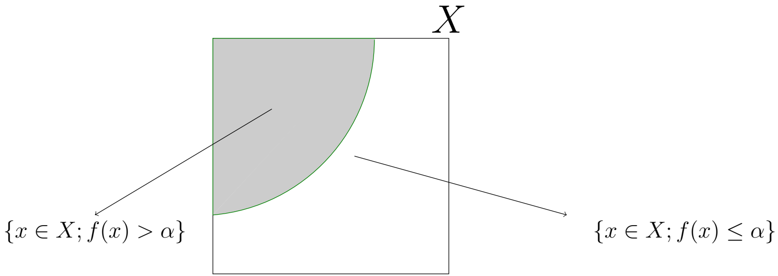
$$\mu(\mathsf{s}_{\mathsf{int}}(f)\langle\mathbf{E}\rangle > \alpha) := \inf\left\{\mu(b) : \; b \in \mathbf{E}_{\mathrm{B}}, \; \underset{X \setminus b}{\mathrm{outsup}}\, \mathsf{s}_\infty(f) \leq \alpha\right\}$$

$$= \inf\{\mu(b) : \; b \in \mathbf{E}_{\mathrm{B}}, \; \sup |f|[b] \leq \alpha\}$$

Is the new concept a generalization of the original one?

For each $\alpha > 0$ we may write

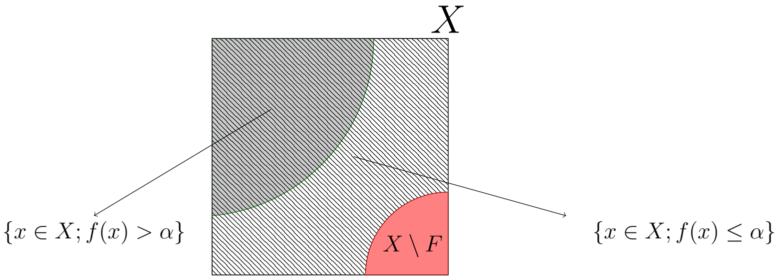
$$h_{\mu,f}(\alpha) = \mu(\{x \in X; |f(x)| > \alpha\}) = \inf \Big\{ \mu(b) : b \in \mathbf{E}_{\mathrm{B}},\ \underbrace{(\forall x \in X \setminus b)\, |f(x)| \leq \alpha}_{\sup |f|[X \setminus b] \leq \alpha} \Big\}.$$



Is the new concept a generalization of the original one?

For each $\alpha > 0$ we may write

$$h_{\mu,f}(\alpha) = \mu(\{x \in X; |f(x)| > \alpha\}) = \inf\Big\{\mu(b) : b \in \mathbf{E}_{\mathrm{B}},\ \underbrace{(\forall x \in X \setminus b)\, |f(x)| \le \alpha}_{\sup |f|[X \setminus b] \le \alpha}\Big\}.$$



How far is the new concept from the old one?

$$\boxed{\mathsf{s}_{\mathsf{sum}}(f)(a) = \sum_{i \in a} |f(i)|}$$

$$\mu(\mathsf{s}_{\mathsf{sum}}(f)\langle\mathbf{E}\rangle > \alpha) = \inf\left\{\mu(b) : \sum_{i \in X \setminus a} |f(i)| \leq \alpha\right\}$$

$$\boxed{?\mu(\mathsf{s}(f)\langle\mathbf{E}\rangle > \alpha) = m(\{x \in X;\ |G_f(x)| > \beta_\alpha\})?}$$

$X = \{a, b, c\}$

We assume that μ is strictly increasing with respect to the following order $\prec$ on $\mathbf{E}_{\text{power}}$:

$$\emptyset \prec \{a\} \prec \{b\} \prec \{c\} \prec \{a, b\} \prec \{a, c\} \prec \{b, c\} \prec X.$$

We define a function f on X as $f(a) = 2$, $f(b) = 3$, $f(c) = 4$.

How far is the new concept from the old one?

$$\boxed{\mathsf{s}_{\mathsf{sum}}(f)(a) = \sum_{i \in a} |f(i)|}$$

$$\mu(\mathsf{s}_{\mathsf{sum}}(f)\langle \mathbf{E} \rangle > \alpha) = \inf \left\{ \mu(b) : \sum_{i \in X \setminus a} |f(i)| \leq \alpha \right\}$$

$$\boxed{?\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = m(\{x \in X;\ |G_f(x)| > \beta_\alpha\})?}$$

$X = \{a, b, c\}$

We assume that μ is strictly increasing with respect to the following order $\prec$ on $\mathbf{E}_{\text{power}}$:

$$\emptyset \prec \{a\} \prec \{b\} \prec \{c\} \prec \{a, b\} \prec \{a, c\} \prec \{b, c\} \prec X.$$

We define a function f on X as $f(a) = 2$, $f(b) = 3$, $f(c) = 4$.

How far is the new concept from the old one?

$$\boxed{\mathsf{s}_{\mathsf{sum}}(f)(a) = \sum_{i \in a} |f(i)|}$$

$$\mu(\mathsf{s}_{\mathsf{sum}}(f)\langle \mathbf{E} \rangle > \alpha) = \inf \left\{ \mu(b) : \sum_{i \in X \setminus a} |f(i)| \leq \alpha \right\}$$

$$\boxed{?\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = m(\{x \in X;\ |G_f(x)| > \beta_\alpha\})?}$$

$X = \{a, b, c\}$

We assume that μ is strictly increasing with respect to the following order $\prec$ on $\mathbf{E}_{\text{power}}$:

$$\emptyset \prec \{a\} \prec \{b\} \prec \{c\} \prec \{a, b\} \prec \{a, c\} \prec \{b, c\} \prec X.$$

We define a function f on X as $f(a) = 2$, $f(b) = 3$, $f(c) = 4$.

How far is the new concept from the old one?

$$\boxed{\mathsf{s}_{\mathsf{sum}}(f)(a) = \sum_{i \in a} |f(i)|}$$

$$\mu(\mathsf{s}_{\mathsf{sum}}(f)\langle\mathbf{E}\rangle > \alpha) = \inf\left\{\mu(b) : \sum_{i \in X \setminus a} |f(i)| \leq \alpha\right\}$$

$$\boxed{?\mu(\mathsf{s}(f)\langle\mathbf{E}\rangle > \alpha) = m(\{x \in X;\ |G_f(x)| > \beta_\alpha\})?}$$

$X = \{a, b, c\}$

We assume that μ is strictly increasing with respect to the following order $\prec$ on $\mathbb{E}_{\text{power}}$:

$$\emptyset \prec \{a\} \prec \{b\} \prec \{c\} \prec \{a, b\} \prec \{a, c\} \prec \{b, c\} \prec X.$$

We define a function f on X as $f(a) = 2$, $f(b) = 3$, $f(c) = 4$.

How far is the new concept from the old one?

$$\boxed{\mathsf{s}_{\mathsf{sum}}(f)(a) = \sum_{i \in a} |f(i)|}$$

$$\mu(\mathsf{s}_{\mathsf{sum}}(f)\langle \mathbf{E} \rangle > \alpha) = \inf \left\{ \mu(b) : \sum_{i \in X \setminus a} |f(i)| \leq \alpha \right\}$$

$$\boxed{?\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = m(\{x \in X;\ |G_f(x)| > \beta_\alpha\})?}$$

$X = \{a, b, c\}$

We assume that μ is strictly increasing with respect to the following order $\prec$ on $\mathbf{E}_{\text{power}}$:

$$\emptyset \prec \{a\} \prec \{b\} \prec \{c\} \prec \{a, b\} \prec \{a, c\} \prec \{b, c\} \prec X.$$

We define a function f on X as $f(a) = 2$, $f(b) = 3$, $f(c) = 4$.

How far is the new concept from the old one?

Topological space X, Borel subsets $\mathbf{E}_\mathrm{B}$, monotone measure $\mu : \mathbf{E}_\mathrm{B} \to [0, +\infty]$

- A new underlying set $\mathbf{E}_\mathrm{B}$.
- A new induced monotone measure $m_\mu : 2^{\mathbf{E}_\mathrm{B}} \to [0, +\infty]$

$$m_\mu(F) := \inf\{\mu(a) : \ a \in \mathbf{E}_\mathrm{B} \setminus F\}.$$

- A new induced function $\mathbf{t}_f : \mathbf{E}_\mathrm{B} \to [0, +\infty]$

$$\mathbf{t}_f(a) := \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle.$$

Proposition

Let $(X, \mathbf{E}, \mathsf{s})$ be a sub-Borel size space. Then for every $f \in \mathcal{B}(X)$ we have

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = m_\mu(\{a \in \mathbf{E}_\mathrm{B} : \ \mathbf{t}_f(a) > \alpha\}).$$

Proof. Just different notation:

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = \inf \left\{ \mu(a) : \ a \in \mathbf{E}_\mathrm{B}, \ \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle \le \alpha \right\} =$$

$$= \inf \{\mu(a) : \ a \in \mathbf{E}_\mathrm{B}, \ \mathbf{t}_f(a) \le \alpha\} = m_\mu(\{a \in \mathbf{E}_\mathrm{B} : \ \mathbf{t}_f(a) > \alpha\}).$$

□

How far is the new concept from the old one?

Topological space X, Borel subsets $\mathbf{E}_\mathrm{B}$, monotone measure $\mu : \mathbf{E}_\mathrm{B} \to [0, +\infty]$

- A new underlying set $\mathbf{E}_\mathrm{B}$.
- A new induced monotone measure $m_\mu : 2^{\mathbf{E}_\mathrm{B}} \to [0, +\infty]$

$$m_\mu(F) := \inf\{\mu(a) : \ a \in \mathbf{E}_\mathrm{B} \setminus F\}.$$

- A new induced function $\mathbf{t}_f : \mathbf{E}_\mathrm{B} \to [0, +\infty]$

$$\mathbf{t}_f(a) := \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle.$$

Proposition

Let $(X, \mathbf{E}, \mathsf{s})$ be a sub-Borel size space. Then for every $f \in \mathcal{B}(X)$ we have

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = m_\mu(\{a \in \mathbf{E}_\mathrm{B} : \ \mathbf{t}_f(a) > \alpha\}).$$

Proof. Just different notation:

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = \inf \left\{ \mu(a) : \ a \in \mathbf{E}_\mathrm{B}, \ \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle \leq \alpha \right\} =$$

$$= \inf \{\mu(a) : \ a \in \mathbf{E}_\mathrm{B}, \ \mathbf{t}_f(a) \leq \alpha\} = m_\mu(\{a \in \mathbf{E}_\mathrm{B} : \ \mathbf{t}_f(a) > \alpha\}).$$

□

How far is the new concept from the old one?

Topological space X, Borel subsets $\mathbf{E}_{\mathrm{B}}$, monotone measure $\mu : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

- A new underlying set $\mathbf{E}_{\mathrm{B}}$.
- A new induced monotone measure $m_\mu : 2^{\mathbf{E}_{\mathrm{B}}} \to [0, +\infty]$

$$m_\mu(F) := \inf\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}} \setminus F\}.$$

- A new induced function $\mathbf{t}_f : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

$$\mathbf{t}_f(a) := \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle.$$

Proposition

Let $(X, \mathbf{E}, \mathsf{s})$ be a sub-Borel size space. Then for every $f \in \mathcal{B}(X)$ we have

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \ \mathbf{t}_f(a) > \alpha\}).$$

Proof. Just different notation:

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = \inf\left\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}, \ \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle \le \alpha\right\} =$$

$$= \inf\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}, \ \mathbf{t}_f(a) \le \alpha\} = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \ \mathbf{t}_f(a) > \alpha\}).$$

□

How far is the new concept from the old one?

Topological space X, Borel subsets $\mathbf{E}_{\mathrm{B}}$, monotone measure $\mu : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

- A new underlying set $\mathbf{E}_{\mathrm{B}}$.
- A new induced monotone measure $m_\mu : 2^{\mathbf{E}_{\mathrm{B}}} \to [0, +\infty]$

$$m_\mu(F) := \inf\{\mu(a) : \; a \in \mathbf{E}_{\mathrm{B}} \setminus F\}.$$

- A new induced function $\mathbf{t}_f : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

$$\mathbf{t}_f(a) := \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle.$$

Proposition

Let $(X, \mathbf{E}, \mathsf{s})$ be a sub-Borel size space. Then for every $f \in \mathcal{B}(X)$ we have

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \; \mathbf{t}_f(a) > \alpha\}).$$

Proof. Just different notation:

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = \inf\left\{\mu(a) : \; a \in \mathbf{E}_{\mathrm{B}}, \; \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle \le \alpha\right\} =$$

$$= \inf\{\mu(a) : \; a \in \mathbf{E}_{\mathrm{B}}, \; \mathbf{t}_f(a) \le \alpha\} = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \; \mathbf{t}_f(a) > \alpha\}).$$

□

How far is the new concept from the old one?

Topological space X, Borel subsets $\mathbf{E}_{\mathrm{B}}$, monotone measure $\mu : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

- A new underlying set $\mathbf{E}_{\mathrm{B}}$.
- A new induced monotone measure $m_\mu : 2^{\mathbf{E}_{\mathrm{B}}} \to [0, +\infty]$

$$m_\mu(F) := \inf\{\mu(a) : \; a \in \mathbf{E}_{\mathrm{B}} \setminus F\}.$$

- A new induced function $\mathbf{t}_f : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

$$\mathbf{t}_f(a) := \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle.$$

Proposition

Let $(X, \mathbf{E}, \mathrm{s})$ be a sub-Borel size space. Then for every $f \in \mathcal{B}(X)$ we have

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \; \mathbf{t}_f(a) > \alpha\}).$$

Proof. Just different notation:

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = \inf\left\{\mu(a) : \; a \in \mathbf{E}_{\mathrm{B}}, \; \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle \le \alpha\right\} =$$

$$= \inf\{\mu(a) : \; a \in \mathbf{E}_{\mathrm{B}}, \; \mathbf{t}_f(a) \le \alpha\} = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \; \mathbf{t}_f(a) > \alpha\}).$$

□

How far is the new concept from the old one?

Topological space X, Borel subsets $\mathbf{E}_{\mathrm{B}}$, monotone measure $\mu : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

- A new underlying set $\mathbf{E}_{\mathrm{B}}$.
- A new induced monotone measure $m_\mu : 2^{\mathbf{E}_{\mathrm{B}}} \to [0, +\infty]$

$$m_\mu(F) := \inf\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}} \setminus F\}.$$

- A new induced function $\mathbf{t}_f : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

$$\mathbf{t}_f(a) := \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle.$$

Proposition

Let $(X, \mathbf{E}, \mathrm{s})$ be a sub-Borel size space. Then for every $f \in \mathcal{B}(X)$ we have

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \ \mathbf{t}_f(a) > \alpha\}).$$

Proof. Just different notation:

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = \inf \left\{ \mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}, \ \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle \le \alpha \right\} =$$

$$= \inf\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}, \ \mathbf{t}_f(a) \le \alpha\} = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \ \mathbf{t}_f(a) > \alpha\}).$$

□

How far is the new concept from the old one?

Topological space X, Borel subsets $\mathbf{E}_{\mathrm{B}}$, monotone measure $\mu : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

- A new underlying set $\mathbf{E}_{\mathrm{B}}$.
- A new induced monotone measure $m_\mu : 2^{\mathbf{E}_{\mathrm{B}}} \to [0, +\infty]$

$$m_\mu(F) := \inf\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}} \setminus F\}.$$

- A new induced function $\mathbf{t}_f : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

$$\mathbf{t}_f(a) := \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle.$$

Proposition

Let $(X, \mathbf{E}, \mathsf{s})$ be a sub-Borel size space. Then for every $f \in \mathcal{B}(X)$ we have

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \ \mathbf{t}_f(a) > \alpha\}).$$

Proof. Just different notation:

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = \inf\left\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}, \ \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle \le \alpha\right\} =$$

$$= \inf\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}, \ \mathbf{t}_f(a) \le \alpha\} = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \ \mathbf{t}_f(a) > \alpha\}).$$

□

How far is the new concept from the old one?

Topological space X, Borel subsets $\mathbf{E}_{\mathrm{B}}$, monotone measure $\mu : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

- A new underlying set $\mathbf{E}_{\mathrm{B}}$.
- A new induced monotone measure $m_\mu : 2^{\mathbf{E}_{\mathrm{B}}} \to [0, +\infty]$

$$m_\mu(F) := \inf\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}} \setminus F\}.$$

- A new induced function $\mathbf{t}_f : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

$$\mathbf{t}_f(a) := \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle.$$

Proposition

Let $(X, \mathbf{E}, \mathrm{s})$ be a sub-Borel size space. Then for every $f \in \mathcal{B}(X)$ we have

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \ \mathbf{t}_f(a) > \alpha\}).$$

Proof. Just different notation:

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = \inf \left\{ \mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}, \ \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle \leq \alpha \right\} =$$

$$= \inf \{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}, \ \mathbf{t}_f(a) \leq \alpha\} = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \ \mathbf{t}_f(a) > \alpha\}).$$

□

How far is the new concept from the old one?

Topological space X, Borel subsets $\mathbf{E}_{\mathrm{B}}$, monotone measure $\mu : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

- A new underlying set $\mathbf{E}_{\mathrm{B}}$.
- A new induced monotone measure $m_\mu : 2^{\mathbf{E}_{\mathrm{B}}} \to [0, +\infty]$

$$m_\mu(F) := \inf\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}} \setminus F\}.$$

- A new induced function $\mathbf{t}_f : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

$$\mathbf{t}_f(a) := \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle.$$

Proposition

Let $(X, \mathbf{E}, \mathrm{s})$ *be a sub-Borel size space. Then for every* $f \in \mathcal{B}(X)$ *we have*

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \ \mathbf{t}_f(a) > \alpha\}).$$

Proof. Just different notation:

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = \inf \left\{ \mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}, \ \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle \le \alpha \right\} =$$

$$= \inf \{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}, \ \mathbf{t}_f(a) \le \alpha\} = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \ \mathbf{t}_f(a) > \alpha\}).$$

□

How far is the new concept from the old one?

Topological space X, Borel subsets $\mathbf{E}_{\mathrm{B}}$, monotone measure $\mu : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

- A new underlying set $\mathbf{E}_{\mathrm{B}}$.
- A new induced monotone measure $m_\mu : 2^{\mathbf{E}_{\mathrm{B}}} \to [0, +\infty]$

$$m_\mu(F) := \inf\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}} \setminus F\}.$$

- A new induced function $\mathbf{t}_f : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$

$$\mathbf{t}_f(a) := \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle.$$

Proposition

Let $(X, \mathbf{E}, \mathsf{s})$ be a sub-Borel size space. Then for every $f \in \mathcal{B}(X)$ we have

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \ \mathbf{t}_f(a) > \alpha\}).$$

Proof. Just different notation:

$$\mu(\mathsf{s}(f)\langle \mathbf{E} \rangle > \alpha) = \inf\left\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}, \ \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle \le \alpha\right\} =$$

$$= \inf\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}, \ \mathbf{t}_f(a) \le \alpha\} = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \ \mathbf{t}_f(a) > \alpha\}).$$

□

How good is the induced measure?

$$m_\mu(F) := \inf\{\mu(a) : \ a \in \mathbf{E}_\mathrm{B} \setminus F\}.$$

Proposition

Let X be a topological space and μ be a monotone measure on $\mathbf{E}_\mathrm{B}$.

(a) m_μ *is superadditive.*

(b) $m_\mu(\bigcap \mathcal{A}) = \inf\{m_\mu(A) : \ A \in \mathcal{A}\}$ *for any* $\mathcal{A} \subseteq 2^{\mathbf{E}_\mathrm{B}}$.

(c) m_μ *is upper semicontinuous.*

How good is the induced measure?

$$m_\mu(F) := \inf\{\mu(a) :\ a \in \mathbf{E}_\mathrm{B} \setminus F\}.$$

Lemma

Let X be a topological space and μ be a monotone measure on $\mathbf{E}_\mathrm{B}$.

(a) *If $\mu(a) = 0$ and $a \notin F$ then $m_\mu(F) = 0$.*

(b) *If $\emptyset \notin F$ then $m_\mu(F) = 0$.*

(c) *If $a \neq \emptyset$ then $m_\mu(\{a\}) = 0$.*

(d) *If $|\mathcal{N}_\mu| > 1$ then $m_\mu(\{\emptyset\}) = 0$.*

(e) *$m_\mu(F) = 0$ if and only if for any $\varepsilon > 0$ there is $a \notin F$ such that $\mu(a) < \varepsilon$.*

(f) *$m_\mu(\mathbf{E}_\mathrm{B}) = \mu(X)$.*

(g) *$\mu(a) = m_\mu(\mathbf{E}_\mathrm{B} \setminus \{a\})$.*

(h) *$m_\mu(F) = \inf\{m_\mu(\mathbf{E}_\mathrm{B} \setminus \{a\}) :\ F \subseteq \mathbf{E}_\mathrm{B} \setminus \{a\}\}$.*

What are properties of the induced function?

$$\mathbf{t}_f(a) := \operatorname*{outsup}_{X \setminus a} \mathsf{s}(f)\langle \mathbf{E} \rangle.$$

Proposition

Let $(X, \mathbf{E}, \mathsf{s})$ be a sub-Borel size space, then for every $f \in \mathcal{B}(X)$ we have

(a) $\mathbf{t}_f(\emptyset) = \sup_{E \in \mathbf{E}} \mathsf{s}(f)(E)$ *and* $\mathbf{t}_f(X) = 0$.

(b) $\mathbf{t}_f$ *is anti-monotone, i.e.,* $\mathbf{t}_f(a_2) \leq \mathbf{t}_f(a_1)$ *whenever* $a_1 \subseteq a_2$.

(c) *If* $a_1, a_2 \in \mathbf{E}_{\mathrm{B}}$ *then* $\mathbf{t}_f(a_1 \cap a_2) \leq C_{\mathsf{s}}(\mathbf{t}_f(a_1) + \mathbf{t}_f(a_2))$.

What is next?

Question

What are properties of the smallest σ-algebra on $\mathbf{E}_{\mathrm{B}}$ such that all $\mathbf{t}_f$ are measurable?

Question

Which topologies on $\mathbf{E}_{\mathrm{B}}$ do make $\mathbf{t}_f$ Borel on $\mathbf{E}_{\mathrm{B}}$?

Question

Are there other transformation methods?

What is next?

Question

What are properties of the smallest σ-algebra on $\mathbf{E}_{\mathrm{B}}$ such that all $\mathbf{t}_f$ are measurable?

Question

Which topologies on $\mathbf{E}_{\mathrm{B}}$ do make $\mathbf{t}_f$ Borel on $\mathbf{E}_{\mathrm{B}}$?

Question

Are there other transformation methods?

What is next?

Question

What are properties of the smallest σ-algebra on $\mathbf{E}_{\mathrm{B}}$ such that all $\mathbf{t}_f$ are measurable?

Question

Which topologies on $\mathbf{E}_{\mathrm{B}}$ do make $\mathbf{t}_f$ Borel on $\mathbf{E}_{\mathrm{B}}$?

Question

Are there other transformation methods?

Thanks for Your attention!